

Instanton representation of Plebanski gravity: IX.
Hamiltonian minisuperspace dynamics in
undensitized momentum space variables

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Abstract

In this paper we illustrate the dynamics of the instanton representation in the description of vacuum GR in minisuperspace for undensitized variables. We uncover a new class of general solutions in both the degenerate and the nondegenerate sectors of the theory. Additionally, the individual sectors are preserved under Hamiltonian evolution. Finally, we present an algorithm for constructing general solutions by expansion about the isotropic sector of the instanton representation.

1 Introduction

We will acquire some intuition regarding the classical dynamics of the instanton representation of Plebanski gravity, starting in this paper with a cursory analysis of minisuperspace dynamics in undensitized momentum space variables. The action for the instanton representation is equivalent to that in the Ashtekar variables when restricted to nondegenerate metrics. In this paper we will examine the dynamics of both the degenerate and the nondegenerate sectors of the theory. The instanton representation on the full phase space Ω_{Inst} is given by the first order phase space action

$$I_{Inst} = \int dt \int_{\Sigma} d^3x \left(\Psi_{ae} B_e^i \dot{A}_i^a + \Psi_{ae} B_e^i D_i A_0^a - N^\mu H_\mu \right). \quad (1)$$

$N^\mu = (N, N^i)$ are respectively the lapse function and the shift vector of general relativity, and $H_\mu = (H, H_i)$ are the Hamiltonian and the diffeomorphism constraints, given by

$$H = (\det B)^{1/2} \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}); \quad H_i = \epsilon_{ijk} B_a^j B_e^k \Psi_{ae}. \quad (2)$$

The object Ψ_{ae} is a $SO(3, C) \times SO(3, C)$ valued matrix known as the CDJ matrix [1], and is of mass dimension $[\Psi_{ae}] = -2$. B_e^i is the magnetic field for a self dual $SO(3, C)$ gauge connection A_i^a ,¹ where

$$B_a^i = \epsilon^{ijk} \partial_j A_k^a + \frac{1}{2} \epsilon^{ijk} f_{abc} A_j^b A_k^c. \quad (3)$$

Under the CDJ Ansatz

$$\Psi_{ae}^{-1} = (\tilde{\sigma}^{-1})_i^a B_e^i, \quad (4)$$

then (1) reduces for $(\det B) \neq 0$ and $\det \Psi \neq 0$ to the action for GR in the Ashtekar variables where $\tilde{\sigma}_a^i$ is the densitized triad (see e.g. [2],[3],[4]). Hence (1) is a formulation of general relativity in which the CDJ matrix Ψ_{ae} is regarded as a fundamental dynamical variable.

To study the minisuperspace dynamics of the instanton representation, we must now reduce (1) to minisuperspace. Minisuperspace is defined as

¹Our notation is that symbols from the beginning of the Latin alphabet a, b, c, \dots denote internal $SO(3, C)$ indices, while from the middle of the alphabet i, j, k, \dots denote spatial indices.

the sector of the full theory where all variables are spatially homogeneous. This means that all spatial gradients must be set to zero, which is unlike the usual definition of minisuperspace which uses Bianchi symmetry groups [5]. Hence in minisuperspace as we have defined it, (3) reduces to

$$B_a^i = (\det A)(A^{-1})_a^i; \quad \det B = (\det A)^2. \quad (5)$$

We have used in (5) the fact that the structure constants f_{abc} for $SO(3, C)$ are numerically the same as the three dimensional epsilon symbol ϵ_{abc} , in writing the determinant. We must now reduce the constraints (2) to minisuperspace. The Hamiltonian constraint H is given by

$$H = (\det A)\sqrt{\det \Psi}(\Lambda + \text{tr} \Psi^{-1}) \quad (6)$$

and the diffeomorphism constraint H_i is given by

$$H_i = (\det A)^2(\det A)^{-1}A_i^d\psi_d = (\det A)A_i^d\psi_d, \quad (7)$$

where $\psi_d = \epsilon_{dbf}\Psi_{bf}$ is derived from the antisymmetric part of Ψ_{bf} . A direct way to obtain the Gauss' Law constraint for minisuperspace is to obtain

$$\Psi_{ae}B_e^iD_iA_0^a = \Psi_{ae}B_e^i(\partial_iA_0^a + f^{abc}A_i^bA_0^c) = \Psi_{ae}(\det A)(A^{-1})_e^if^{abc}A_i^bA_0^c, \quad (8)$$

and then vary (8) with respect to A_0^a , yielding

$$G_a = (\det A)\psi_a. \quad (9)$$

In minisuperspace the both the Gauss' law and the diffeomorphism constraints depend linearly on ψ_d , and are therefore redundant.

The last remaining object needed is the canonical structure which determines the canonical one form, which is given by

$$\dot{X}^{ae} = B_e^i\dot{A}_i^a = (\det A)(A^{-1})_e^i\dot{A}_i^a. \quad (10)$$

While the velocity \dot{X}^{ae} is defined by (10), the issue of the existence of X^{ae} as a global coordinate on configuration space Γ_{Inst} arises.² As we will show, the existence or nonexistence of X^{ae} is not relevant as far as the dynamics are concerned, since we will be able to formulate the Hamilton's equations of motion using only the velocities \dot{X}^{ae} without actually making use of X^{ae} .

²The Soo one forms δX^{ae} were first introduced in [6] and [7].

The organization of this paper is as follows. Section 2 treats the nondegenerate case, computing the general solution to the equations of motion, and section 3 derives the induced time development of the spacetime metric. Sections 4 and 5 recompute the equations of motion for the degenerate case, and construct some new solutions. Section 6 introduces a method for constructing a general solution by expansion about the isotropic sector.

2 Setting the stage with nondegenerate vacuum general relativity ($X = \det A \neq 0$)

The action for general relativity in the instanton representation in minisuperspace with Lorentzian signature is given by

$$I_{Inst} = \int_0^T dt \left(-i\Psi_{ae}\dot{X}^{ae} + (\det A) \left[iN\sqrt{\det\Psi}(\Lambda + \text{tr}\Psi^{-1}) + (N^i A_i^d - \theta^d)f_{dbf}\Psi_{bf} \right] \right), \quad (11)$$

The Hamilton's equations of motion for \dot{X}^{ae} are given by

$$\dot{X}^{ae} = B_e^i \dot{A}_i^a = \frac{\delta H}{\delta \Psi_{ae}} = i \left[\frac{1}{2}(\Psi^{-1})^{ae} H[N] - N(\det A)\sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{ae} \right] + (\det A)(N^i A_i^d - \theta^d)f_{dae}; \quad (12)$$

and for the CDJ matrix Ψ_{ae} by

$$\dot{\Psi}_{ae} = -\frac{\delta H}{\delta X^{ae}} = - \left[i\delta_{ae}(\det A)^{-1}H[N] + \delta_{ae}(N^i A_i^d - \theta^d)f_{dbf}\Psi_{bf} \right]. \quad (13)$$

Since the Cauchy development of GR should be consistent with the initial value constraints, then the constraints can be applied wherever they appear in the equations of motion. For $\det B = (\det A)^2 \neq 0$ the only constraint contained in (12) is the Hamiltonian constraint $H \sim 0$.³ On the other hand, all terms of (13) are directly proportional to constraints.

The diffeomorphism constraint $H_i = 0$ implies that $f_{dbf}\Psi_{bf} = 0$, or $\Psi_{bf} = \Psi_{(bf)}$ is symmetric in bf . Application of this and the Hamiltonian constraint $H = 0$ in (13) yields

$$\dot{\Psi}_{ae} = 0 \rightarrow \Psi_{ae}(t) = \lambda_{(ae)} = \text{const.} \quad \forall a, e. \quad (14)$$

However, for (12), only the Hamiltonian constraint can be used. The solution to the Hamiltonian constraint is given by

$$\lambda_3 = -\frac{\lambda_1 \lambda_2}{\Lambda \lambda_1 \lambda_2 + \lambda_1 + \lambda_2}. \quad (15)$$

³We have used the identity $B_a^i = (\det A)(A^{-1})_a^i$, as well as $X = \text{tr} X^{ae} = \det A$, for anisotropic minisuperspace.

To obtain some physical insight into the configuration space dynamics, let us take the symmetric part of (12), which has the same effect as making a gauge choice $N^i = \theta^d = 0$. Then (12) and (13) reduce to

$$\dot{X}^{(ae)} = -iN(\det A)\eta^{ae}; \quad \dot{\Psi}_{ae} = 0, \quad (16)$$

where we have defined an internal $SU(2)_- \otimes SU(2)_-$ metric η^{ae} (not to be confused with the Minkoski metric η^{ij} which has spatial indices i, j), by

$$\eta^{ae} = \sqrt{\det \Psi}(\Psi^{-1}\Psi^{-1})^{ae}. \quad (17)$$

Equation (16) resembles the equations of motion for a free particle in classical mechanics, travelling ostensibly through a nine dimensional configuration space. However, the space is actually five dimensional when restricted to the constraint surface defined by the initial value constraints.

Taking the trace of (16) and dividing through by $X = \text{tr} X^{ae} = \det A$ since $X \neq 0$ due to nondegeneracy, we obtain⁴

$$\frac{\dot{X}}{X} = \frac{d \ln X}{dt} = -i\eta N \quad (18)$$

where we have defined $\eta = \delta_{ae}\eta^{ae}$ as the trace. Since X is globally a holonomic coordinate on Γ_{Inst} , then equation (18) directly integrates to

$$X(t) = (\det A(t)) = X_0 e^{-i\eta \int_0^t N(t') dt'} \quad (19)$$

where $X_0 = X(0)$. Note the mass dimension $[\eta] = 1$ which cancels the negative mass dimension of time $[t] = -1$ so that the argument of the exponential is dimensionless. Substituting (19) back into (16) we obtain the equation for the velocity \dot{X}^{ae}

$$\dot{X}^{(ae)} = -iN\eta^{ae} \left(X_0 e^{-i\eta \int_0^t N(t') dt'} \right) = \frac{\eta^{ae}}{\eta} \frac{d}{dt} \left(X_0 e^{-i\eta \int_0^t N(t') dt'} \right). \quad (20)$$

The right hand side of (20) is a total time derivative since according to (14) and (17), η^{ae} is constant in time. This implies that for the equation to make sense the left hand side must as well be a total derivative. However, there is no coordinate X^{ae} on configuration space, since $\delta X^{ae} = B_e^i \delta A_i^a \not\subset \wedge^a(\Gamma_{Inst})$ is not an exact one form except for its trace. However, there exist

⁴Note that X is actually the Chern–Simons functional for minisuperspace, identified in [7] as the candidate for a time variable on configuration space in quantum cosmology.

configurations for which a densitized X^{ae} can in some sense be defined. For instance, for diagonal connections $A_i^a = \delta_i^a A_a^a$ one has

$$X = (\det A) = A_1^1 A_2^2 A_3^3; \quad B_1^1 = A_2^2 A_3^3; \quad B_2^2 = A_3^3 A_1^1; \quad B_3^3 = A_1^1 A_2^2 \quad (21)$$

and (16) reduces to

$$A_2^2 A_3^3 \dot{A}_1^1 = -iN\eta^{11}(A_1^1 A_2^2 A_3^3) \longrightarrow \frac{\dot{A}_1^1}{A_1^1} = -iN\eta^{11}, \quad (22)$$

and likewise for A_2^2 and A_3^3 . Then (22) integrates to

$$A_f^f(t) = A_f^f(0) \left(\frac{X(t)}{X(0)} \right)^{\eta^{ff}/\eta}, \quad (23)$$

for $f = 1, 2, 3$. The result is that all components of the connection evolve with respect to X , seen as a time variable on configuration space.

2.1 Dynamics of the spacetime metric

Since the instanton representation is a metric-free description of gravity, then the spacetime metric $g_{\mu\nu}$ is a derived quantity. The lapse function $N(t)$, can still be chosen arbitrarily, but the spatial 3-metric h_{ij} is determined dynamically through the evolution of the instanton representation phase space. The spatial three metric h^{ij} is given by

$$h^{ij} = \frac{\Psi_{ae}\Psi_{af}}{\det\Psi} \left(\frac{B_e^i B_f^j}{\det B} \right) = \frac{\tilde{\sigma}_a^i \tilde{\sigma}_a^j}{\det\tilde{\sigma}}. \quad (24)$$

The relation to η_{ae} in (17) stems more directly from the covariant form

$$h_{ij} = (\det\Psi)(\Psi^{-1}\Psi^{-1})^{ef}(B^{-1})_i^e(B^{-1})_j^f(\det B), \quad (25)$$

whereupon the following relation can be written

$$\eta^{ae} = \sqrt{\det\Psi}(\Psi^{-1}\Psi^{-1})^{ae} \equiv (\det\Psi)^{-1/2} h_{ij} \left(\frac{B_a^i B_e^j}{\det B} \right). \quad (26)$$

Making use of the minisuperspace relations $B_a^i = (\det A)(A^{-1})_a^i$ and $\det B = (\det A)^2 = X^2$, and $\sqrt{\det\Psi} = (\det\eta)^{-1}$, then (25) can be written as

$$h_{ij} \sim h_{ij}(t; \lambda) = (\det\eta)^{-1} \eta^{ae}(\lambda) A_i^a(t) A_j^e(t). \quad (27)$$

Observe that h_{ij} has acquired the label of $\eta^{ae} \in GL(5, C)$, and also depends on A_i^a .⁵ The part of h_{ij} which can always be unambiguously specified is $h = \det(h_{ij}) = (\det\eta)^{-2}X^2$, whose time evolution is given by

$$\sqrt{h(t)} = (\det\eta)^{-1}X(0)e^{-i\eta \int_0^t N(t')dt'}. \quad (28)$$

Hence in the general solution for the nondegenerate case even h , acquires the label of $\lambda_{ae} \in GL(5, C)$. One could then construct a spacetime metric

$$ds^2 = -N^2 dt^2 + (\det\eta)^{-1}\eta^{ae}\omega^a \otimes \omega^e, \quad (29)$$

whence the Ashtekar potential A_i^a becomes absorbed into the definition of the one forms ω^i , given by

$$\omega^a = A_i^a(dx^i + N^i dt). \quad (30)$$

While (30) resembles the invariant one forms defined for Bianchi groups [5], this is not the case since A_i^a does not satisfy the Maurer–Cartan equation since it is not a flat connection.

For the configuration (23) one obtains the time evolution for the spatial 3-metric

$$h_{ij}(t) = \delta_{if}\delta_{jf}(\det\eta)^{-1}\eta^{ff}(A_f^f(0))^2 \left(\frac{X(t)}{X(0)}\right)^{2\eta^{ff}/\eta}, \quad (31)$$

which also evolves in relation to $X(t)$, seen as a time variable on configuration space. This can be written explicitly in terms of metric variables as

$$h_{ij}(t) = \delta_{if}\delta_{jf}h_{ff}(0) \left(\frac{h(t)}{h(0)}\right)^{\eta^{ff}/\eta}, \quad (32)$$

where now the determinant of the 3-metric plays the role of the time variable, and the components of the metric evolve with respect to it. From (32) one finds the physical interpretation of the components of the connection A_f^f in terms of h_{ij} , which fixes its value at $t = 0$ to

$$h_{ff}(0) = (\det\eta)^{-1}\eta^{ff}(A_f^f(0))^2. \quad (33)$$

⁵According to the previous section, the information encoding the evolution of specific components of A_i^a , is tied up in the combination $\delta X^{ae} = B_e^i \delta A_i^a$. The trace of X^{ae} , namely the Chern–Simons invariant, undergoes a well-defined evolution (19) from which some information regarding $A_i^a(t)$ can be inferred.

To obtain a real section of GR, one imposes reality conditions on the instanton representation variables so that the 3-metric is real. For example, we must have η^{ff}/η real for each f , namely that

$$Im(\eta^{11}\eta^{22}) = Im(\eta^{22}\eta^{33}) = Im(\eta^{33}\eta^{11}) = 0. \quad (34)$$

The real parts of η^{11} , η^{22} and η^{33} and one imaginary part, say $Im(\eta^{33})$ are freely specifiable, which fixes $Im(\eta^{11})$ and $Im(\eta^{22})$. In order for the metric to be real at $t = 0$, then (33) in turn requires that each component of the connection be either pure real or pure imaginary.

3 The degenerate sector ($\det A=0$)

Having obtained a solution for the nondegenerate sector, let us now examine the degenerate sector ($\det A = 0$). This implies that the metric is also degenerate, since $h = (\det A)^2 (\det \Psi)$. While this may be the case, as we will show, the time evolution of the CDJ matrix is still well-defined. Recall the expression for the Hamiltonian constraint

$$H = (\det A) \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}), \quad (35)$$

which includes $\det A$ as part of its definition, therefore when $\det A = 0$ (35) is trivially satisfied with no restrictions on Ψ_{ae} . This means that the CDJ matrix Ψ_{ae} is now free to evolve in time. Let us now revisit the equations of motion (12) and (13) under the condition of degeneracy. Starting with the equation for X^{ae} , requoting (12) for completeness,

$$\begin{aligned} \dot{X}^{ae} = B_e^i \dot{A}_i^a = i \left[\frac{1}{2} (\Psi^{-1})^{ae} H[N] \right. \\ \left. - N (\det A) \sqrt{\det \Psi} (\Psi^{-1} \Psi^{-1})^{ae} \right] + (\det A) (N^i A_i^d - \theta^d) f_{dae}. \end{aligned} \quad (36)$$

Since $\det A = 0$, then (36) the trace of (36) vanishes. This implies that the trace X is numerically constant, and since $X(t) = \det A = 0 = X_0$, it remains degenerate for all times. Therefore, the degenerate case remains a distinct sector of vacuum GR which cannot be bridged.⁶

Moving on to the equation of motion for Ψ_{ae} , we have from (13) that

$$\dot{\Psi}_{ae} = -i \delta_{ae} N \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}) - \delta_{ae} (N^i A_i^d - \theta^d) f_{dbf} \Psi_{bf}. \quad (37)$$

The Gauss' law and the diffeomorphism constraints imply that $\Psi_{bf} = \Psi_{(bf)}$ is symmetric in b, f , therefore the second term of (37) vanishes and we are left with the equations

$$\dot{\Psi}_{ae} = -i \delta_{ae} N \sqrt{\det \Psi} (\Lambda + \text{tr} \Psi^{-1}); \quad \dot{X}^{ae} = 0. \quad (38)$$

Comparison of (38), corresponding to $(\det A) = 0$ with the analogous equations (16) for $(\det A) \neq 0$ reveals the following contrast. Whereas for $(\det A) \neq 0$ the momentum variable Ψ_{ae} was constant in time while \dot{X}^{ae} was nontrivial, we see for $(\det A) = 0$ that it is \dot{X}^{ae} which vanishes while

⁶Hence, the configuration space variables cannot evolve in time from the $X = 0$ sector into the $X \neq 0$ sector, since the equations of motion are assumed to hold for all time. Consequently, topology change is precluded, unlike in [11] and works by other authors.

Ψ_{ae} inherits a nontrivial time evolution. The ‘configuration’ and the ‘momentum’ space have essentially ‘exchanged’ roles as a consequence of the degeneracy condition.

Equation (38) states that the time derivative of Ψ_{ae} is an isotropic matrix, which means that the off-diagonal parts are numerical constants. Since $\Psi_{[ae]} = 0$ on account of the kinematic constraints, we may assume that Ψ_{ae} is a symmetric matrix of the form

$$\Psi_{ae}(t) = \begin{pmatrix} a(t) & W & V \\ W & b(t) & U \\ V & U & c(t) \end{pmatrix}$$

where U , V and W are arbitrary numerical constants. Without loss of generality we can set $U = V = W = 0$, since Ψ_{ae} can always be diagonalized if it is nondegenerate. Note that these initial conditions are preserved on account of the equations of motion, and the nontrivial equations of motion then reduce to

$$\dot{a} = \dot{b} = \dot{c} = iN\sqrt{abc}\left(\Lambda + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right). \quad (39)$$

The time derivatives of the diagonal elements are equal, which implies that these elements must be equal, within numerical constants, to each other

$$b(t) = a(t) + k_1; \quad c(t) = a(t) + k_2 \quad (40)$$

for arbitrary constants k_1 and k_2 . In the general case the equation of motion can be integrated

$$\int_{a_0}^{a(t)} (a(a+k_1)(a+k_2))^{-1/2} \left(\Lambda + \frac{1}{a} + \frac{1}{a+k_1} + \frac{1}{a+k_2}\right)^{-1} da = i \int_0^t N(t') dt'. \quad (41)$$

The left hand side of (41) can be written in closed form in terms of known functions, but we do not display the result it here. Let us rather use a compact notation to describe the integral

$$I_\Lambda(a; k_1, k_2) = i \int_0^t N(t') dt' = \tau(t). \quad (42)$$

One can then in principle invert (42) to find the evolution of $a = a(\tau)$ as a function of the ‘time’ τ , which depends on the choice of lapse function N and is labelled by the constants k_1 and k_2 .

3.1 A few simple cases within the diagonal sector

Let us now illustrate a few simple examples for the degenerate case for which short expressions can be written.

Case (i): Isotropic case with $\Lambda \neq 0$. In this case we have $k_1 = k_2 = 0$. The equation of motion reduces to

$$\dot{a} = iNa^{3/2}\left(\Lambda + \frac{3}{a}\right) \quad (43)$$

Equation (43) directly integrates to

$$I_\Lambda(a; 0, 0) = \tau(t); \quad a(\tau) = \frac{3}{\Lambda} \tan\left[\tan^{-1}\left(\sqrt{\frac{\Lambda a_0}{3}} + \frac{\sqrt{3\Lambda}}{2}\tau\right)\right] \quad (44)$$

For $\Lambda = 0$ we have

$$a(\tau) = \left(\sqrt{a_0} + \frac{3}{2}\tau\right)^2. \quad (45)$$

Case (ii): One diagonal degree of freedom. In this case $k_2 = 0$, $k_1 = k$ where k is an arbitrary numerical constant. Then we have

$$\dot{a} = iNa\sqrt{a+k}\left(\Lambda + \frac{2}{a} + \frac{1}{a+k}\right) \quad (46)$$

Equation (46) integrates to

$$I_\Lambda(a; k, 0) = \sqrt{\frac{2}{\Lambda}} \left[\frac{r_- \tanh^{-1}\left(\frac{\sqrt{2\Lambda(a+k)}}{r_-}\right) - r_+ \tanh^{-1}\left(\frac{\sqrt{2\Lambda(a+k)}}{r_+}\right)}{k\Lambda - 3 - r_-^2} \right] = \tau \quad (47)$$

where we have defined the dimensionless constants

$$r_- = \sqrt{k\Lambda - 3 - \sqrt{k^2\Lambda^2 - 2k\Lambda + 9}}; \quad r_+ = \sqrt{k\Lambda - 3 + \sqrt{k^2\Lambda^2 - 2k\Lambda + 9}}. \quad (48)$$

While solved in closed form, (49) is nontrivial to invert for $a = a(t)$, but nevertheless the solution is implicit. The $\Lambda = 0$ case leads to the relation

$$I_0(a; k, 0) = \frac{2}{3} \left[\sqrt{a+k} - \sqrt{\frac{k}{3}} \tanh^{-1}\left(\sqrt{3\left(1 + \frac{a}{k}\right)}\right) \right] = \tau. \quad (49)$$

Likewise, the relation (49) is implicit but still nevertheless illustrates the integrability of the system.

Case (iii): General diagonal case for $\Lambda = 0$. This is given by

$$I_0(a; k_1, k_2) = \int_{a_0}^{a(t)} \frac{\sqrt{a(a+k_1)(a+k_2)}}{3a^2 + 2a(k_1 + k_2) + k_1 k_2} da = \tau \quad (50)$$

The relation (50) can as well be integrated in closed form in terms of known functions, though we do not display the final expression here.

4 The degenerate general case

We now treat the general solution for the degenerate case, solving the equation of motion

$$\dot{\Psi}_{ae} = -i\delta_{ae}N\sqrt{\det\Psi}(\Lambda + \text{tr}\Psi^{-1}); \quad \dot{X}^{ae} = 0 \quad (51)$$

for a more general form of Ψ_{ae} . This is given by

$$\Psi_{ae} = \begin{pmatrix} a & W & V \\ W & b & U \\ V & U & c \end{pmatrix}.$$

It will be convenient to parametrize Ψ_{ae} by its isotropic and its non-isotropic parts, as in

$$\Psi_{ae} = \delta_{ae}\varphi + \epsilon_{ae}. \quad (52)$$

In (52), φ is the isotropic part and ϵ_{ae} is defined as the ‘CDJ deviation matrix’. Substitution of (52) into (51) yields

$$\dot{\epsilon}_{ae} = 0; \quad \epsilon_{ae} = \text{const.} \quad \forall a, e \quad (53)$$

The deviation matrix ϵ_{ae} is in general an 8 by 8 matrix of arbitrary complex numerical constants.

Let us now re-examine the equation of motion for Ψ_{ae} . This can be written as one equation labelled by the constants $\epsilon_{ae} \in GL(8, C)$.

$$\dot{\varphi} = -iN\sqrt{\det(\delta_{ae}\varphi + \epsilon_{ae})}(\Lambda + \text{tr}(\delta_{ae}\varphi + \epsilon_{ae})^{-1}). \quad (54)$$

To solve the equation of motion (54) we will need the determinant, given by

$$\det\Psi_{ae} = \det(\delta_{ae}\varphi + \epsilon_{ae}) = \varphi^3 + \varphi^2\epsilon + \frac{1}{2}\varphi\text{Var}\epsilon + \det\epsilon \quad (55)$$

where we have defined

$$\epsilon = \delta_{ae}\epsilon_{ae} = \text{tr}(\epsilon_{ae}); \quad \text{Var}\epsilon = (\delta_{bf}\delta_{cg} - \delta_{cf}\delta_{bg})\epsilon_{bf}\epsilon_{cg} = (\text{tr}\epsilon)^2 - \text{tr}\epsilon^2. \quad (56)$$

We will also need the inverse of (52). Let us assume for simplicity that the isotropic part φ is the largest component.⁷ Hence, $\varphi > \epsilon_{ae} \quad \forall a, e$, with $\varphi \neq 0$. Then we have

⁷The trace is the dynamical component and will vary in time. Therefore, the expansion is good only for $t = \tau$ such that $\varphi(\tau) > \max\{\epsilon_{ae}\}$. Since the non-isotropic components are numerical constants, then $\forall t$ such that $\varphi(t) > \epsilon_{ae} \quad \forall a, e$ do not hold, then a singularity exists and one can must attempt to find the solution for different ϵ_{ae} . Hence we have shown that singularities can develop only when considering degenerate configurations.

$$\Psi_{ae}^{-1} = (\delta_{ae}\varphi + \epsilon_{ae})^{-1} = \varphi^{-1} \delta_{aa_0} \left(\sum_{n=0}^{\infty} (-1)^n \varphi^{-n} \epsilon_{a_0 a_1} \epsilon_{a_1 a_2} \cdots \epsilon_{a_{n-1} a_n} \right) \delta_{a_n e} \quad (57)$$

The trace of (57) is given by

$$\text{tr} \Psi^{-1} = \varphi^{-1} \sum_{n=0}^{\infty} (-1)^n \varphi^{-n} \epsilon_{aa_1} \epsilon_{a_1 a_2} \cdots \epsilon_{a_{n-1} a} \quad (58)$$

Likewise, the square root of the determinant can be expanded as in

$$\sqrt{\det \Psi} = \varphi^{3/2} \left(1 + \frac{\epsilon}{\varphi} + \frac{\text{Var} \epsilon}{2\varphi^2} + \frac{\det \epsilon}{\varphi^3} \right)^{1/2} \quad (59)$$

Then one can use the infinite binomial series expansion

$$\left(1 + \frac{\epsilon}{\varphi} + \frac{\text{Var} \epsilon}{2\varphi^2} + \frac{\det \epsilon}{\varphi^3} \right)^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)n! 2^{2n}} \left(\frac{\epsilon}{\varphi} + \frac{\text{Var} \epsilon}{2\varphi^2} + \frac{\det \epsilon}{\varphi^3} \right)^n. \quad (60)$$

We must now put (58) and (60) back into (54).

4.1 Asymptotic expansion about the isotropic sector

We are now ready to write down the general solution for the degenerate case to all orders. The equation of motion for φ can be written as

$$\left(\Lambda + \frac{3}{\varphi} \right) \varphi^{-3/2} \dot{\varphi} = -iN' E(\varphi, \epsilon) \quad (61)$$

where we have defined a ‘correction’ factor for non-isotropy E

$$E(\varphi, \epsilon) = \left[1 - \varphi^{-2} \left(\Lambda + \frac{3}{\varphi} \right)^{-1} I_1(\varphi, \epsilon_{ae}) \right] I_3(\varphi, \epsilon_{ae}), \quad (62)$$

where

$$\begin{aligned} I_1(\varphi, \epsilon_{ae}) &= \delta_{a_0} \sum_{n=1}^{\infty} (-1)^n \varphi^{-n} \epsilon_{aa_1} \epsilon_{a_1 a_2} \cdots \epsilon_{a_{n-1} a_n} \delta_{a_n, e}; \\ I_3(\varphi, \epsilon_{ae}) &= \left(1 + \frac{\epsilon}{\varphi} + \frac{\text{Var} \epsilon}{2\varphi^2} + \frac{\det \epsilon}{\varphi^3} \right)^{1/2} \end{aligned} \quad (63)$$

The left hand side of (61) reduces to the isotropic case considered in (43). Upon integration, we have

$$\int_0^t \left(\Lambda + \frac{3}{\varphi} \right) \varphi^{-3/2} d\varphi = -i \int_0^t N(t') E(\varphi(t'), \epsilon) dt' \quad (64)$$

which can be written in the form

$$\tan^{-1} \left(\sqrt{\frac{\Lambda \varphi(t)}{3}} \right) = \tan^{-1} \left(\sqrt{\frac{\Lambda \varphi_0}{3}} \right) + \frac{\sqrt{3\Lambda}}{2} \int_0^t E(\varphi(t'), \epsilon) dt'. \quad (65)$$

Equation (65) further simplifies to

$$\sqrt{\varphi(t; \epsilon)} = \left(\sqrt{\varphi_0} + \frac{3}{\Lambda \sqrt{\varphi_0}} \right) \left[1 - \sqrt{\frac{\Lambda \varphi_0}{3}} \tan \frac{\sqrt{3\Lambda}}{2} \int_0^t E(\varphi; \epsilon) dt' \right]^{-1} - \frac{3}{\Lambda \sqrt{\varphi_0}}. \quad (66)$$

We can now generate an asymptotic expansion about the exact isotropic solution by fixed point iteration procedure, assuming that the right hand side of (64) is small. First, choose a particular state ϵ_{ae} , thought of as a constant vector in an eight complex dimensional space. Next, define a sequence of functions of time $\varphi_n(t)$, such that $\varphi = \varphi_0$, which is the initial value of the isotropic part of the CDJ matrix at time $t = 0$.⁸ Define sequence φ_n , such that

$$\varphi_{n+1}(t) = \sqrt{\frac{3}{\Lambda}} \tan \left[\tan^{-1} \left(\sqrt{\frac{\Lambda \varphi_0}{3}} \right) + \frac{\sqrt{3\Lambda}}{2} \int_0^t E(\varphi_n(t'), \epsilon) dt' \right]. \quad (67)$$

The full solution, assuming convergence of the sequence (67), is given by $\varphi(t) = \lim_{n \rightarrow \infty} \varphi_n(t)$. Hence, if it is indeed the case that there was a phase of the universe in which degenerate metrics exist, (i) The universe cannot transition into a nondegenerate state. (ii) The existence of such a configuration is labelled by eight arbitrary complex constants for the ϵ_{ae} , each of which defines a state of the universe. (iii) The convergence of the sequence (67) is a necessary, but not sufficient condition for the well-definedness of this configuration.⁹ This in turn depends upon the initial value of φ_0 in relation to the constants ϵ_{ae} . The theorist has free choice of ϵ_{ae} , within the range of parameters yielding a fixed point. However, φ_0 is determined by the initial conditions of the universe, which can only be extrapolated from the cosmological data in existence today.

⁸This time labels the beginning of the universe, and φ_0 is an essential parameter of the theory.

⁹There is clearly a large category of cosmological evidence that dictates that the space-time metric today is nondegenerate.

5 Conclusion

We have illustrated some of the dynamics of gravity in the instanton representation in minisuperspace for undensitized CDJ matrix Ψ_{ae} , seen as a momentum space variable. Starting from this representation, which is defined independently of the existence of a spacetime metric, we have derived solutions which include a class of inflating spacetimes labelled by the eigenvalues of Ψ_{ae} . We have also demonstrated that the big bang singularity can be avoided in this representation independently of any quantum corrections by applying the equations of motion to arbitrarily small times, including the beginning of the universe when quantum effects are presumably dominant. We have also illustrated the dynamics of the theory in the degenerate sector, displaying an array of new solutions. Another result of the initial value constraints development in these variables is that an initially degenerate solution will remain degenerate for all times, which eliminates topology changing configurations classically.

The present paper has demonstrated the anisotropic minisuperspace sector of the model. We have also provided a prescription for obtaining general solutions by asymptotic expansion about an isotropic solution. It is hoped that this algorithm can be of use as well in numerical treatments of general relativity. While the configuration space variables X^{ae} which would ordinarily be conjugate to Ψ_{ae} are not globally integrable for generic configurations, there is certain information which can be inferred from specific combinations of these variables which corresponds to a well-defined time evolution. A future paper along these lines will investigate the structure of X^{ae} , using a version of the instanton representation where the variables are densitized, and as well carry out the analogous computations of the present paper.

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